

# Hagedorn temperature and physics of black holes

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A mini-review devoted to some implications of the Hagedorn temperature for black hole physics. The existence of a limiting temperature is a generic feature of string models. The Hagedorn temperature was introduced first in the context of hadronic physics. Nowadays, the emphasis is shifted to fundamental strings which might be a necessary ingredient to obtain a consistent theory of black holes. The point is that, in field theory, the local temperature close to the horizon could be arbitrarily high, and this observation is difficult to reconcile with the finiteness of the entropy of black holes. After preliminary remarks, we review our recent attempt to evaluate the entropy of large black holes in terms of fundamental strings. We also speculate on implications for dynamics of large- $N_c$  gauge theories arising within holographic models.

## INTRODUCTION.

### From hadrons to black holes

This talk was presented at a session devoted to the 50th anniversary of the introduction of the Hagedorn temperature [1]. Originally, the Hagedorn temperature  $T_H$  was discussed in connection with hadronic physics. Alternatively, one can say that it referred to a mass scale of the order of the pion mass,  $T_H \sim m_\pi$ . We will discuss the physics of black holes, or, more precisely, the properties of the black-hole horizon. In field theory, the local temperature close to the horizon can be arbitrarily high, and this is known to be inconsistent with finiteness of the Bekenstein-Hawking entropy of black holes, see, in particular, [2][3]. The inconsistency becomes manifest at the gravitational scale, or at  $T \sim M_{\text{Planck}}$ . This observation serves as a motivation [4] to introduce strings at this scale. It is within this framework that we make our remarks on the explicit evaluation of the Bekenstein-Hawking entropy, see [5] and references therein. Nowadays, the literature on the subject of strings and black holes is huge. Because of the format of the talk we limit our list of references to only a few papers. General background can be found in [6].

It should be remarked that the recent firewall paradox (supposedly burning up infalling observers) has rekindled the interest in horizon physics (see [7] and subsequent work). We however are interested in the physics as described by static observers and are hence (a priori) safely away from speculating on the experience of infalling observers.

The outline of the talk is as follows. In the Introduction we describe briefly how the notion of a limiting temperature arises within a generic string picture. In Section 2 we remind the reader of the basics of the black holes. In Section 3 we address the issue of evaluat-

ing black-hole entropy within string theory. In Section 4 we discuss briefly possible phenomenological implications within holographic models.

### Hagedorn temperature

As a starting point, we can choose the assumption made by R. Hagedorn [1] that the density of hadronic states  $\omega(E)$  at large energy  $E$  grows exponentially:

$$\omega(E) \sim \exp(\beta_H E) \text{ , where } \beta_H \sim m_\pi^{-1}. \quad (1)$$

Then the partition function

$$z = \int_0^\infty dE \omega(E) e^{-\beta E} \quad (2)$$

exists only as far as  $\beta > \beta_H$ . In the microcanonical language, there is a “limiting temperature”  $T_H$ :

$$\frac{1}{T} = \frac{\partial S}{\partial E} \quad (3)$$

and

$$T < T_H \text{ , } T_H \equiv \frac{1}{\beta_H} = \text{const} \text{ .} \quad (4)$$

The physics behind this remarkable phenomenon is actually quite simple: if we pump energy into the system, new higher-mass states are produced rather than that the energy of already existing states is increased. The increasing of energy of existing states would mean an increasing temperature. The dominance of production of new massive states manifests the emergence of a limiting temperature.

## Hagedorn temperature and strings

In view of the fact that the spectrum (1) leads to such a drastic conclusion as the existence of a limiting temperature in nature, we should probably re-examine the reasons for introducing the exponential spectrum itself.

The strongest support for the assumption (1) comes from the string model of hadrons. And, in turn, the strongest point of the string model is that it reproduces linear Regge trajectories. Indeed, the energy of a string of length  $L$  is given by:

$$E_{\text{string}} = \sigma \cdot L, \quad \sigma \equiv (2\pi\alpha')^{-1}, \quad (5)$$

where  $\sigma$  is the string tension. For a rotating string

$$M^2 = \frac{1}{\alpha'} J, \quad (6)$$

where  $J$  is the total angular momentum. In other words, the Regge trajectories are linear. Moreover, the density of states is indeed exponential at high energy:

$$\omega(E) \sim \frac{\exp(\beta_H E)}{E^{1+D/2}}, \quad (7)$$

where  $\beta_H \sim \sqrt{\alpha'}$  and  $D$  is the number of (non-compact) spatial directions. Derivations of Eqs (5), (6), (7) can be found in standard textbooks.

### Limiting temperature vs phase transition

At first sight, the argumentation above looks strong enough and we could expect the existence of a limiting temperature. In fact, it was realized a long time ago that there is a viable alternative to the introduction of the Hagedorn temperature. Namely, one can argue that there is a phase transition. While at low temperatures hadrons appear to be fundamental they are in fact composite and are built up by quarks and gluons. Thus, at some critical temperature  $T_{cr}$  there is a phase transition to deconfinement. This phase transition has been observed and studied thermodynamically in great detail through lattice simulations for various non-Abelian gauge groups, including the realistic case of quantum chromodynamics.

The precise relation between  $T_H$  and  $T_{cr}$  remains somewhat obscure. Phenomenologically, it is obvious that one should have  $T_H > T_{cr}$ , where  $T_H$  is defined within the string model. How close  $T_H$  is to  $T_{cr}$ , remains unclear because of the uncertainties of the string models of hadrons.

Thus, we can say that the existence of  $T_{cr}$  can be traced back to the fact that it is the field theory which is fundamental, not the hadronic strings model.

As we will see next, the quantum field theory (QFT) becomes, in turn, problematic at the gravitational scale. This is revealed by considering black holes.

## BLACK HOLES. PRELIMINARIES

Let us first remind the reader a few well-known equations concerning black holes. The Schwarzschild geometry reads as

$$ds^2 = -\left(1 - \frac{2G_N M}{r}\right) dt^2 + \left(1 - \frac{2G_N M}{r}\right)^{-1} dr^2 + d\mathbf{x}_\perp^2, \quad (8)$$

where  $M$  is the mass of the central body, or black hole in our case and  $G_N$  is the Newton constant. Note that the  $G_{00}$  component of the metric vanishes at the horizon  $r_H = 2G_N M$ .

The thermodynamic entropy is proportional to the area of the black hole:

$$S_{BH} = \frac{\text{Area}}{4G_N} \quad (9)$$

and there is Hawking radiation with temperature

$$\beta_{\text{Hawking}} = 8\pi G_N M. \quad (10)$$

Close to the horizon, it is useful to introduce the distance  $\rho$  to the horizon,  $\rho = \sqrt{8G_N M(r - 2G_N M)}$ . Then for  $\rho \ll 4G_N M$

$$ds_{\text{Rindler}}^2 = -\frac{\rho^2}{(4G_N M)^2} dt^2 + d\rho^2 + d\mathbf{x}_\perp^2.$$

Many results apply just in this limit of the so-called Rindler space. For Euclidean time  $\tau$ ,

$$ds_{\text{Euclidean}}^2 = \frac{\rho^2}{(4G_N M)^2} d\tau^2 + d\rho^2 + d\mathbf{x}_\perp^2,$$

which is flat space in polar coordinates. Moreover, the  $\tau$ -coordinate is periodic,

$$\tau \sim \tau + \beta_{\text{Rindler}}, \quad (11)$$

with  $\beta_{\text{Rindler}} = 8\pi G_N M \equiv \beta_{\text{Hawking}}$ .

One can say that a black hole provides a “lab” to study temperatures arbitrarily high. Indeed, near the horizon the blue-shift factor is given by

$$\chi \equiv \frac{4G_N M}{\rho}, \quad (12)$$

where  $\rho$  is the distance to the horizon. Hence

$$\beta_{\text{local}} = \beta_{\text{Rindler}} \chi^{-1}, \quad \beta_{\text{local}} \rightarrow 0, \quad \text{if } \rho \rightarrow 0. \quad (13)$$

Note that if we go beyond the Rindler approximation the overall Euclidean thermal manifold is cigar-shaped.

### Black holes and limiting temperature

In quantum field theory the entropy density  $s \sim T^3$  and by using (12) the total entropy stored near the horizon is estimated as

$$S \sim \int d\rho T(\rho)^3 = \frac{\text{Area}}{\epsilon^2}, \quad (14)$$

where  $\epsilon$  is a cut off at small distances.

Not to exceed the black hole entropy (9), we need a limiting temperature (brick wall of 't Hooft [2][3]). In other words, there is a need for a modification of QFT at short distances. Strings are welcome back on the fundamental level!

In practice, to limit applicability of field theory near the horizon one introduces a so-called stretched horizon. The stretched horizon is a surface placed close to the actual horizon, in front of it, such that  $G_{00} \ll 1$ . For more details see [8], [9].

## STRINGY HORIZON

### Long-string picture of L. Susskind

Consider the formation of a black hole by throwing in matter focused inside the (future) black hole. For a distant observer, the matter falls in infinitely long. As a result, the infalling matter spreads out in the transverse directions [4]. Indeed, consider the parton-model representation of the matter. Then there is diffusion of the partons in the transverse directions. And since the process takes long, the partons cover the whole area. Moreover, it is known that, say, two long strings merge into a single one, because of entropic considerations. In this way one comes to the long-string picture of L. Susskind. According to this picture near the horizon, at  $\rho \sim l_s$  where  $l_s$  is the string scale, there exists a single long string<sup>1</sup>.

Moreover, one might hope that by counting the number of states of the long string, one could reproduce the entropy (9) of the black hole:

$$S_{\text{long string}} = S_{BH} \quad (?) \quad (15)$$

Note, though, that the matching (15) is not without problems. Indeed, the density of states of a long string is exponential in its length, or mass  $M$ , see (7). On the other hand, the Bekenstein-Hawking entropy is proportional to the area of the black hole, or its mass squared,  $S_{BH} \sim M^2$ . To maintain (15) one is forced to speculate that this apparent mismatch is removed by accounting for the self-gravitation of the long string [10].

Thus, the long-string picture has definite advantages, by resolving the ultraviolet divergence (12) through the introduction of a finite  $l_s$  and by relating the number of degrees of freedom of a black hole to the number of degrees of freedom of a long string (which is much better understood). Also, the proportionality of the entropy to

the area comes out naturally because of the random walk of partons in the transverse directions.

Many questions are left open, however. In particular:

- What keeps the long string at  $\rho \sim l_s$ ?
- How to get quantitatively  $S = (\text{Area})/4G_N$ ?
- Qualitative picture vs fundamental strings?

These questions were addressed in [5], see also references therein.

## Main results

We considered a kind of mean field approximation, when a thin shell of matter of mass  $\delta M$  falls into the black hole of mass  $M$ . The gravitational field of the mass  $M$  is taken into account while the self-interaction within the shell is neglected. The thin shell is described in the framework of string theory. We find that in Euclidean time the shell occupies a zero mode. This allows us to evaluate the entropy carried to the black hole by the shell. Integrating over  $dM$  reproduces the Bekenstein-Hawking entropy for black holes. Moreover, knowing the wave function of the zero mode allows us to visualize the profile of the stringy horizon at distances of order  $l_s$  from the horizon. One can summarize the results by saying that, in the approximation of the mean field, the idea of relating the entropy of (large) black holes to the stringy degrees of freedom is realized on a fully quantitative level. An important reservation is that these results hold in the case of type II superstrings and heterotic strings in Rindler space. Whereas for bosonic strings, there arises no consistent picture.

## Main tool: thermal scalar

On the technical side, the results just mentioned are obtained by utilizing the construction of the so-called thermal scalar in curved space, see [11] and references therein. The theory of the thermal scalar has been developing since long, see in particular [12], [13]. Roughly speaking, the thermal scalar is a Euclidean counterpart of the Hagedorn transition. Namely, the original mechanism for the Hagedorn divergence of the partition function (2) is the production of high-mass states. A complementary view on the Hagedorn transition in Euclidean time is that it is a kind of a Higgs phase transition when the mass squared of a complex field changes its sign. This scalar field, or thermal scalar, lives in spatial dimensions only (not temporal) and its mass is given by:

$$m_{\text{thermal scalar}}^2 = \frac{(\beta - \beta_H)\beta_H}{2\pi(\alpha')^2}. \quad (16)$$

<sup>1</sup> It is worth emphasizing that physical pictures in general relativity are observer-dependent. The infalling matter is visualized as a single long string by a distant observer.

Eq. (16) holds in flat space and as far as  $m_{\text{thermal scalar}}^2$  is positive. What happens at  $\beta < \beta_H$  is not clear a priori, for further discussion see [13].

The equivalence between the standard formulation of the Hagedorn transition and that one in terms of the thermal scalar can readily be demonstrated using the polymer, or random-walk formulation of Euclidean field theory. It is also straightforward to see that the thermal scalar corresponds to a string once wrapped around the compact Euclidean time [12]. The time dependence is fixed then by the periodicity and effectively the wave function of the thermal scalar depends only on the spatial coordinates.

To consider strings in the background field of black holes, one needs to generalize the thermal scalar to curved space, see [14], [11] and references therein.

### Thermal scalar in curved space

The action for the thermal-scalar field  $\varphi$  is given by [11, 15]:

$$S = \int d^{D-1}x \sqrt{G} e^{-2\Phi} \times \left( G_{ij} \nabla^i \varphi \nabla^j \varphi^* + \frac{1}{4\pi^2(\alpha')^2} (\beta^2 G_{00} - \beta_H^2) \varphi \varphi^* \right), \quad (17)$$

where  $G_{00}, G_{ij}$  are the components of the metric and  $\Phi$  is the dilaton field. A crucial point is that for type II strings in Rindler space,  $ds_{\text{Rindler}}^2 = a^2 \rho^2 dt^2 + d\rho^2 + d\mathbf{x}_\perp^2$ , the action (17) receives no  $\alpha'$  corrections [11, 15]. Moreover, this is also true for heterotic strings [16] (but not for bosonic strings).

The corresponding equation for the wave functions

$$\left( -\partial_\rho^2 - \frac{1}{\rho} \partial_\rho + \frac{1}{4\pi^2(\alpha')^2} (\beta^2 a^2 \rho^2 - \beta_H^2) \right) \varphi_n(\rho) = \lambda_n \varphi_n(\rho) \quad (18)$$

has solutions:

$$\varphi_n(\rho) = \exp\left(-\frac{a\beta\rho^2}{4\pi\alpha'}\right) L_n\left(\frac{a\beta\rho^2}{2\pi\alpha'}\right), \quad \lambda_n = a\beta(1+2n) - 2\pi \quad (19)$$

in terms of Laguerre polynomials  $L_n$ , and  $n \in \mathbb{N}$ . The zero mode ( $n=0$ ) at  $\beta = \beta_{\text{Rindler}} = 2\pi/a$  dominates the thermal partition function.

This analysis was done for the singly wound thermal scalar state. A curiosity at this point is that states that are wound multiple times are simply absent from the spectrum. This will be important in the final section.

### Picture emerging

The build-up of a black hole by throwing a thin shell of mass  $\delta M$  to the black hole of mass  $M_{\text{initial}}$  can be

consistently described by string theory in a mean-field approximation. The shell ends up as a long string in a layer of thickness  $\delta\rho \sim l_s$  near the horizon.

Moreover, the entropy of black holes is calculable without any adjustable parameters. In more detail, the density of states seen by a distant observer is

$$\omega(\delta M) \sim \frac{\exp(\beta_{\text{Hawking}} \delta M)}{\delta M}, \quad \text{or } \beta_{\text{Hawking}} = \beta_{\text{Hagedorn}}. \quad (20)$$

Integrating over  $\delta M$  we get the Bekenstein-Hawking entropy:

$$\delta S_{BH} = 8\pi G_N M \delta M \rightarrow S_{BH} = \frac{\text{Area}}{4G_N}. \quad (21)$$

Also, knowledge of the wave function (19) allows us to evaluate [5, 17] the profile of the energy-momentum tensor associated with the long string of mass  $\delta M$ :

$$\begin{aligned} \langle T_0^0(\mathbf{x}) \rangle &= -2N^2 \left( \frac{2\rho^2}{\alpha'} - 1 \right) e^{-\rho^2/\alpha'}, \\ \langle T_\rho^\rho(\mathbf{x}) \rangle &= 2N^2 e^{-\rho^2/\alpha'}, \\ \langle T_j^i(\mathbf{x}) \rangle &= 2N^2 \delta_j^i \left( 1 - \frac{\rho^2}{\alpha'} \right) e^{-\rho^2/\alpha'}, \end{aligned} \quad (22)$$

where  $i, j = 1, 2$  are transverse directions and  $N$  is a normalization factor.

Note a positive value of the radial pressure,  $\langle T_\rho^\rho(\mathbf{x}) \rangle$ . It is this pressure that keeps the matter from collapsing onto the center. Remarkably, Eqs (22) specify a distribution of matter near the horizon, at  $\rho \sim l_s$ .

### FROM STRINGS TO GAUGE THEORIES, VIA HOLOGRAPHY

Amusingly enough, lessons from strings on the gravitational scale might produce a new insight into the dynamics of gauge theories. The means is holography: strings live in curved extra dimensions, while gauge theory lives on a flat boundary. Both theories are inter-related.

The most famous case of the string-gauge duality refers to  $\mathcal{N} = 4$  supersymmetric Yang-Mills theory [18]. In case of non-supersymmetric Yang-Mills theories, Witten constructed a model [19] which in the far infrared limit belongs to the same universality class as large- $N_c$  gauge theories. The model can be generalized to incorporate massless quarks [20]. From our perspective, it is crucial that the geometry in extra dimensions inherent to this model is of the same cigar-shape as we encountered while discussing large black holes above.

In more detail, the Euclidean version of the model [19] introduces two compact dimensions:

- Periodic Euclidean time  $\tau \sim \tau + \beta_\tau(z)$  where the periodicity,  $\beta_\tau$  depends on an extra coordinate

$z$ . As is common to holographic models, the  $z$ -coordinate is associated with resolution. The limit  $z \rightarrow 0$  corresponds to Yang-Mills theories in the ultraviolet limit, while  $z \rightarrow z_{\text{horizon}}$  corresponds to the infrared limit on the field theoretic side

- There is another periodic coordinate  $\sigma \sim \sigma + \beta_\sigma(z)$ . Wrapping around  $\sigma$  counts the topological charge associated with the corresponding stringy state

From first principles, at  $T = 0$  the  $(\tau, z)$  space is a cylinder and  $(\sigma, z)$  is cigar-shaped,  $\beta_\sigma(z_{\text{horizon}}) = 0$ . At the deconfining phase transition,  $T = T_{cr}$  the geometries in the  $(\tau, z)$  and  $(\sigma, z)$  coordinates are interchanged.

Now we are coming to a central point, namely how, if at all, the geometry in the extra dimensions is related to gauge theory phenomenology. One of the routes is to consider properties of so-called defects. One of the best studied examples of a defect on the field-theoretic side are instantons. On the stringy side, defects can be identified topologically, in terms of wrapping around compact directions. In the geometric language the string action is very simple,

$$(\text{action}) \sim L \cdot (\text{tension}) .$$

If there is a cylinder-type geometry then the wrapping number is well-defined and the probability to find a defect is suppressed by the action.

However, in case of a cigar-shaped geometry,

$$\beta_\tau(z_{\text{horizon}}) = 0 \text{ or } \beta_\sigma(z_{\text{horizon}}) = 0$$

the action for wrapped states vanishes at the horizon and the probability to find them in the vacuum state is not suppressed by their action.

Instantons are distinguished by a non-vanishing topological charge. As is mentioned above, in the geometric language, the topological charge is associated with wrapping around the  $\sigma$ -coordinate. The simplest geometric object which can be wrapped around the  $\sigma$  direction is a  $D0$ -brane. The corresponding defect is characterized by a topological charge determined by the wrapping number, its position in the Euclidean space and by its action, which depends on the value of its  $z$ -coordinate. Thus, the number of the parameters characterizing such  $D0$  branes matches instantons of field theory, for further details and references see [21].

Let us check the gravity-gauge correspondence on the example of the instantons [21]. Consider first temperature  $T = 0$  and start with the strings. Since  $\beta_\sigma$  vanishes on the horizon,  $\beta_\sigma(z = z_{\text{horizon}}) = 0$ , instantons are not suppressed by the action in the far infrared. This is, indeed, well known on the field theoretic side of the correspondence. At  $T = T_{cr}$  the  $(\sigma, z)$  geometry is changed into a cylinder and instantons become suppressed according to the stringy picture. Again, this conclusion is well

known in field theory and is supported by the existing phenomenology.

Now we are coming to the central point of this section, that is the manifestation of the zero mode discussed in the preceding section. Classically, the action for the defects,

$$S_{\text{defect}} = L \cdot (\text{tension})$$

vanishes for any wrapping number as far as  $\beta_\sigma(z) = 0$ . However, quantum-mechanically keeping a  $D0$  brane at the tip of the cigar results in kinetic energy, because of the uncertainty principle. As a result, only the lowest level survives as the zero mode at the tip of the cigar (see discussion in the preceding section). The lowest level, in turn, corresponds to a single wrapping. Phenomenologically, this implies that only instantons with the topological charge  $Q_{\text{top}} = \pm 1$  exist while  $|Q_{\text{top}}| \geq 2$  are suppressed in the vacuum. Also, an effective infrared cut off might arise dynamically in the far infrared. These predictions which follow from holography seem to be supported by the lattice data.

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